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Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "New Family of Modal Methods for Calculating Eigenvector Derivatives"

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THE author of Ref. 1 presented a numerical method for calculating eigenvector derivatives for a linear eigenvalue problem. However, it appears that the left eigenvector and the exact third right eigenvector derivative given in the singular nonself-adjoint example are in error. This makes all results doubtful.

The exact first and the second eigenvalue and eigenvector derivatives are calculated for this example using the results in Ref. 2 which gives the general exact analytical solutions for nth derivatives of simple (unrepeated) eigenvalues and eigenvectors for a general linear and nonlinear eigenvalue problems. These exact solutions are always more accurate than approximate solutions obtained from various numerical methods as shown in Ref. 3 for a nonlinear example. In general, the existence of the exact analytical solutions obviates the need for using any numerical method. Thus, for

$$A = \begin{bmatrix} 1 & g & -1 & 1 & 0 & 0 \\ -1 & -0.8 & g & g & g+1 & g-1 \\ 0 & -1 & 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

where $g = 2.4$.

The eigenvectors are given here with full machine precision, which is 14 significant digits,

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}; \qquad x_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} -1.6 \\ 1 \\ 0 \\ 0.8 \\ -1.3 \\ 0.5 \end{bmatrix}; \qquad x_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1.4 \\ 0.9 \\ 0.5 \end{bmatrix}$$

$x_5 =$

$$\begin{bmatrix} -2.261136712749622e-01 - 5.725627134310787e-01i \\ 1 \\ 6.589861751152075e-01 + 6.814357875308333e-01i \\ -1.292780337941628e+00 + 1.073065550479584e-01i \\ 9.425499231950851e-01 - 6.477556279172403e-01i \\ 3.502304147465438e-01 + 5.404490728692817e-01i \end{bmatrix}$$

$x_6 =$

$$\begin{bmatrix} -2.261136712749622e-01 + 5.725627134310787e-01i \\ 1 \\ 6.589861751152075e-01 - 6.814357875308333e-01i \\ -1.292780337941628e+00 - 1.073065550479584e-01i \\ 9.425499231950851e-01 + 6.477556279172403e-01i \\ 3.502304147465438e-01 - 5.404490728692817e-01i \end{bmatrix}$$

and the left eigenvectors are

$$y_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}; \qquad y_2 = \begin{bmatrix} -1.666666666666652e-01 \\ 6.666666666666645e-01 \\ 2.266666666666666e+00 \\ 6.133333333333327e+00 \\ 5.299999999999994e+00 \\ 2.100000000000001e+00 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 1.282051282051283e-01 \\ 3.846153846153849e-01 \\ -1.051282051282053e+00 \\ -2.230769230769234e+00 \\ -2.358974358974362e+00 \\ -9.230769230769242e-01 \end{bmatrix}$$

$$y_4 = \begin{bmatrix} 7.142857142857143e-01 \\ 7.142857142857141e-01 \\ 7.142857142857140e-01 \\ 2.428571428571428e+00 \\ 1.714285714285714e+00 \\ 1.428571428571429e+00 \end{bmatrix}$$

$y_5 =$

$$\begin{bmatrix} -4.212454212454211e-01 - 1.167357947251096e-01i \\ -4.945054945054921e-02 - 4.762820424784466e-01i \\ 1.684981684981679e-01 + 4.669431789004389e-02i \\ -9.890109890109866e-02 - 9.525640849568934e-01i \\ 3.223443223443223e-01 - 8.358282902317837e-01i \\ -2.527472527472531e-01 - 7.004147683506565e-02i \end{bmatrix}$$

$y_6 =$

$$\begin{bmatrix} -4.212454212454211e-01 + 1.167357947251096e-01i \\ -4.945054945054921e-02 + 4.762820424784466e-01i \\ 1.684981684981679e-01 - 4.669431789004389e-02i \\ -9.890109890109866e-02 + 9.525640849568934e-01i \\ 3.223443223443223e-01 + 8.358282902317837e-01i \\ -2.527472527472531e-01 + 7.004147683506565e-02i \end{bmatrix}$$

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Also, the eigenvalues are

$$\lambda = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1.600000000000000e+00 + 1.019803902718556e+00i \\ 1.600000000000000e+00 - 1.019803902718556e+00i \end{bmatrix}$$

The left and the right eigenvectors satisfy $\mathbf{Y}^{\dagger}X = I$, where I is a 6×6 unity matrix. The matrix Y given in the paper is incorrect since it gives $Y^{\dagger}X \neq I$.

From Ref. 2 the first ith unrepeated eigenvalue and eigenvector derivatives with respect to a parameter $\zeta = g$ are

$$\lambda_i^{(1)} = y_i^{\dagger} A_{\zeta} x_i; \qquad i = 3, 4, 5, 6 \tag{1}$$

$$x_i^{(1)} = x_i \lambda_i^{(1)} - Q_i A_{\zeta} x_i \tag{2}$$

respectively, and the second derivatives are

$$\lambda_i^{(2)} = y_i^{\dagger} z_2 \tag{3}$$

$$x_i^{(2)} = -\left(x_i^{\dagger(1)} x_i^{(1)}\right) x_i + x_i \lambda_i^{(2)} - Q_i z_2 \tag{4}$$

where

$$z_2 = 2\left(A_{\zeta} - \lambda_i^{(1)} I\right) x_i^{(1)} \tag{5}$$

and

$$Q_i = (A + x_i x^{\dagger}{}_i)^{-1} \tag{6}$$

is a nonsingular matrix.

Using these exact solutions given above the first and the second derivatives of unrepeated eigenvalues λ_i and eigenvectors $x_i (i = 3, 4, 5, 6)$ are calculated as

$$\lambda_i^{(1)} = \begin{bmatrix} 1.282051282051283e-01 \\ 1.428571428571428e+00 \\ -7.783882783882780e-01 - 3.969017020653730e-01i \\ -7.783882783882780e-01 + 3.969017020653730e-01i \end{bmatrix}$$

$$\lambda_i^{(2)} = \begin{bmatrix} -7.670392285776921e-02 \\ 4.373177842565563e-01 \\ -1.803069306993958e-01 - 1.417432853638815e+00i \\ -1.803069306993958e-01 + 1.417432853638815e+00i \end{bmatrix}$$

$$x_3^{(1)} = \begin{bmatrix} -2.252150672346113e-01 \\ -3.960995573373426e-01 \\ -6.410256410256414e-02 \\ 1.318383028480749e-01 \\ 2.108911718032125e-03 \\ -1.339472145661071e-01 \end{bmatrix}$$

$$x_4^{(1)} = \begin{bmatrix} -7.066919791172279e-01 \\ 5.790223065970573e-01 \\ 1.293308020882770e+00 \\ 3.322259136212624e-01 \\ -1.336022781205505e+00 \\ 1.003796867584243e+00 \end{bmatrix}$$

$$x_5^{(1)} = \begin{bmatrix} 8.147090902820018e-01 - 8.178712894837928e-01i \\ 2.797207257649567e-01 + 2.013077103594879e-02i \\ 2.414720779309265e-02 - 4.315073179500531e-01i \\ 7.896659986716059e-02 + 2.162307715109216e-01i \\ -5.439247941168346e-02 + 2.527777745630033e-02i \\ -2.457412045547747e-02 - 2.415085489672215e-01i \end{bmatrix}$$

$$x_6^{(1)} = \begin{bmatrix} 8.147090902820018e-01 + 8.178712894837928e-01i \\ 2.797207257649567e-01 - 2.013077103594879e-02i \\ 2.414720779309265e-02 + 4.315073179500531e-01i \\ 7.896659986716059e-02 - 2.162307715109216e-01i \\ -5.439247941168346e-02 - 2.527777745630033e-02i \\ -2.457412045547747e-02 + 2.415085489672215e-01i \end{bmatrix}$$

$$x_3^{(2)} = \begin{bmatrix} 2.274674229140999e-01 \\ 2.324526872685722e-01 \\ 5.626084616312639e-02 \\ -9.938251099961047e-02 \\ 4.763529097749752e-02 \\ 5.174722002211303e-02 \end{bmatrix}$$

$$x_4^{(2)} = \begin{bmatrix} 1.713257658777025e+00 \\ -1.710364281693518e+00 \\ 1.376285660675453e+00 \\ 2.741313167467403e+00 \\ -3.952372805724307e+00 \\ 1.211059638256905e+00 \end{bmatrix}$$

$$x_5^{(2)} = \begin{bmatrix} -1.023534778365032e+00 - 1.846822398918317e+00i \\ -4.020919939672313e-01 - 4.603724148212065e-01i \\ -1.166693328930620e+00 - 5.290360749432439e-01i \\ 7.414098923185688e-01 + 8.639575600068872e-02i \\ -7.022788530496998e-02 + 1.463356430563412e-01i \\ -6.711820070135985e-01 - 2.327313990570288e-01i \end{bmatrix}$$

$$x_6^{(2)} = \begin{bmatrix} -1.023534778365032e+00 + 1.846822398918317e+00i \\ -4.020919939672313e-01 + 4.603724148212065e-01i \\ -1.166693328930620e+00 + 5.290360749432439e-01i \\ 7.414098923185688e-01 - 8.639575600068872e-02i \\ -7.022788530496998e-02 - 1.463356430563412e-01i \\ -6.711820070135985e-01 + 2.324313990570288e-01i \end{bmatrix}$$

The eigenvalues and eigenvectors must satisfy the eigenvalue equation

$$(A - \lambda_i I) x_i = 0; \qquad i = 3, 4, 5, 6 \tag{7}$$

and normalization condition

$$x_i^{\dagger} x_i = 1 \tag{8}$$

and their derivatives must satisfy the derivatives of these equations. The first derivatives of Eqs. (7) and (8) are respectively

$$\left(A^{(1)} - \lambda_i^{(1)} I\right) x_i + \left(A - \lambda_i I\right) x_i^{(1)} = 0 \tag{9}$$

$$x_i^{\dagger} x_i^{(1)} = 0 \tag{10}$$

and the second derivatives are

$$\left(A^{(2)} - \lambda_i^{(2)} I\right) x_i + 2\left(A^{(1)} - \lambda_i^{(1)} I\right) x_i^{(1)} + (A - \lambda_i I) x_i^{(2)} = 0 \tag{11}$$

$$x_i^{\dagger(1)} x_i^{(1)} + x_i^{\dagger} x_i^{(2)} = 0 \tag{12}$$

The left-hand side of Eqs. (7–12) are of the order 10^{-14} which is the full machine accuracy.

These equations are not satisfied using the only solutions given in the paper, which is for the first derivatives of the third eigenvector and eigenvalue. It is very likely that the derivatives for other eigenvalues and eigenvectors would be in error if calculated using the method in Ref. 1.

References

[1] Akgün, M. A., "New Family of Modal Methods for Calculating Eigenvector Derivatives," *AIAA Journal*, Vol. 32, No. 2, 1994, pp. 379–386.

[2]Jankovic, M. S., "Exact *n*th Derivatives of Eigenvalues and Eigenvectors," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 1, 1994, pp. 136–144.

[3]Jankovic, M. S., "Comments on 'Eigenvalue Sensitivity in the Stability Analysis of Beck's Column with a Concentrated Mass at the Free End'," *Journal of Sound and Vibration*, Vol. 167, No. 3, 1993, pp. 557–559.

Reply by the Author to M. S. Jankovic

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JANKOVIC[1] states that the left eigenvectors of the example nonself-adjoint system are in error. Strictly speaking, they are not in error. In other words, they are valid eigenvectors. However, $\{y_2\}$ is not biorthogonal to $\{x_1\}$. This is a good example of a case where special care is required in assuring that the eigenvectors corresponding to a multiple eigenvalue are (bi)orthogonal. The wrong choice of the second left eigenvector was an oversight by the author and the author thanks Jankovic for noticing the error in the computed eigenvector derivative. In Ref. 2, $\{y_2\}$ not being biorthogonal to $\{x_1\}$ led to a wrong value for a_{jk2}, Eq. (16). The exact value of the derivative of the third right eigenvector at the bottom of page 384 was computed with $n = -1$, i.e., the so-called modal method, with the full set of eigenvectors. Since, both the exact value and the values for various n with a truncated set of eigenvectors (top of page 385) used the same incorrect value of a_{jk2}, the latter converged to the former, which was itself in error, and as a result, the mistake went unnoticed. This mistake, however, does not in any way affect the conclusions about the performance of the family of modal methods developed in the paper as will be illustrated next. The matrix $[A]$ and the matrix of right eigenvectors $[X]$ will not be repeated here and can be looked up in the paper or in Jankovic's Comment.[1] The $\{y_j\}$ for $j \neq 2$ need not be repeated either. The newly selected $\{y_2\}$ is

$$\{y_2\}^T = [-0.5 \quad 2 \quad 6.8 \quad 18.4 \quad 15.9 \quad 6.3]$$

In the paper,[2] most of the eigenvectors were computed with Eispack routines. They are computed again here with Matlab this time as was done by Jankovic. Their biorthogonality is checked, and it is observed that $\{y_j\}^T\{x_i\}$ is of the order of 10^{-14} for $i \neq j$. $[D_E]$ is unchanged with the new choice of $\{y_2\}$. Again $\lambda_{3,g} = 10/78$, and the exact derivative of the third right eigenvector, computed with $n = -1$ with the full set of eigenvectors, i.e., with $\hat{N} = N$, comes out to be

$$\{x_3\}_{,g} = \begin{bmatrix} -2.252150672346110e\text{-}01 \\ -3.960995573373423e\text{-}01 \\ -6.410256410256399e\text{-}02 \\ 1.318383028480748e\text{-}01 \\ 2.108911718032278e\text{-}03 \\ -1.339472145661070e\text{-}01 \end{bmatrix}$$

which agrees with Jankovic's solution to the full machine precision.

The same derivative is now computed with higher order methods by using the first four modes as in the original paper.[2] The columns of the matrix given next show, from left to right, the derivative computed with n = 0–7, respectively.

$n = 0$	1	2	3	4	5	6	7
−0.2526	−0.2308	−0.2126	−0.2348	−0.2201	−0.2270	−0.2250	−0.2249
−0.3845	−0.3467	−0.4432	−0.3679	−0.4081	−0.3933	−0.3953	−0.3976
0.0794	−0.1102	−0.0630	−0.0523	−0.0749	−0.0578	−0.0667	−0.0635
0.0363	0.1246	0.1648	0.1046	0.1469	0.1260	0.1328	0.1326
0.0170	0.0333	−0.0298	0.0218	−0.0065	0.0043	0.0025	0.0011
−0.0533	−0.1579	−0.1350	−0.1263	−0.1404	−0.1303	−0.1354	−0.1337

The derivative values computed with n = 12 and 20 are, on the other hand,

$n = 12$	$n = 20$
−2.2521e-01	−2.252153e-01
−3.9614e-01	−3.960991e-01
−6.4060e-02	−6.410227e-02
1.3184e-01	1.318378e-01
2.0840e-03	2.109269e-03
−1.3392e-01	−1.339470e-01

where the number of significant digits used are different to point out the convergence to the exact value. In practice, the 20th-order method would never be used. But then, the example problem, which is of size 6, would be solved with $n = -1$ by using the full set of eigenvectors because 1) the magnitude of the third eigenvalue not being very small compared to the modulus of the fifth and sixth eigenvalues requires a high-order method (i.e., a large value of n) for convergence when the fifth and sixth eigenvectors are not used, and 2) the computational saving, if any, with the use of a high-order method is insignificant for such a small system. The example was intended to demonstrate the convergence of the method for a system with a qualitative relative moduli trend of eigenvalues as described. For large practical systems, however, where the eigenvalues associated with the eigenvector derivatives of interest are much smaller in magnitude then most of the system eigenvalues, a low-order method with a small subset of the eigenvectors is sufficient for convergence. And 14 digit accuracy is hardly required in practical engineering analysis. Jankovic is to be commended for his exact solution of eigenvector derivatives. His assertion[1] that "the existence of exact analytical solutions obviates the need for using any numerical method," however, should be taken with some reser-

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